JOURNAL OF MATERIALS SCIENCE 23 (1988) 2424-2428

# A new modified Weibull distribution function for the evaluation of the strength of silicon carbide and alumina fibres

K. K. PHANI
*Central Glass and Ceramic Research Institute, Calcutta 700032, India*

The strength distributions of silicon carbide and alumina fibres have been evaluated using a modified Weibull distribution function. The function provides an upper and lower strength limit and is characterized by two shape and location parameters. The sum of squares was used as a measure of fit between the distribution function and the data. The result showed good agreement between the two. In addition, the strength distribution and the average value at a different gauge length were extrapolated from the parameters estimated at the original gauge length. In this case also, the proposed function accurately predicted the data points.

## 1. Introduction

The advanced ceramic fibres such as boron, carbon, silicon carbide and alumina are characterized by a variable strength which is usually attributed to the pre-existing flaws of variable sizes in these materials. The statistical distribution nature of these flaws also increases the probability of encountering a more severe flaw with an increase in length, with the consequent reduction in fracture strength in long lengths. Statistically, the analysis of strength is carried out in terms of the "weakest-link theory" which is based on the theory that the fracture is controlled by the weakest defect present in a fibre. The form of weakest-link theory most frequently applied to reinforcing fibres such as silicon carbide, alumina is that due to Weibull [1]. However, as demanded by the theory, the strength distribution usually does not yield a straight line on a Weibull probability graph [2–6]. Also, as opposed to the prediction of the theory, the log strength against log length plot shows a change in slope at short fibre lengths [2, 4]. As an alternative approach, a multi-modal Weibull distribution function [2] is usually used for the analysis of the strength distribution of ceramic fibres, from the viewpoint that the distribution is controlled by more than one type of flaw population arising out of "surface" or "volume" defects. The same approach has been used by Goda and Fukunaga [2] to analyse the strength of two types of advanced metal-reinforcing fibres i.e. silicon carbide and alumina. For silicon carbide fibres they identified two types of defects as "flaw" or "pit" type surface defects and "void" type volume defects. In addition there were undetectable defects which were assigned to the surface. Although the analysis of the strength distribution was carried out assuming these two modes, both the modes contained strength values arising out of failures due to volume or surface defects.

In this paper, statistical justification for such an analysis has been analysed and a modification to the Weibull distribution function has been suggested for the analysis of fibre strength data. The strength distributions of silicon carbide and alumina fibres have also been analysed in terms of the proposed distribution function.

## 2. Analytical procedure

### 2.1. Flaw distribution function

Statistically, for random and independent distributions of flaws the failure probability $P$, of a fibre of length $L$, having a strength less than $S$, is given by [7]

$$P(S) = 1 - \exp[-LN(S)] \quad (1)$$

where $N(S)$ is the $LN(S)$ is the failure probability of one flaw at strength less than $S$. Weibull [1] assumed a reasonable form for the cumulative flaw distribution function as

$$N(S) = (S/S_0)^m \quad (2)$$

where $S_0$ is a scaling parameter and $m$ is a shape parameter. Equations 1 and 2 give

$$P(S) = 1 - \exp[-L(S/S_0)^m] \quad (3)$$

or

$$\ln\{(1/L)\ln[1/(1-P)]\} = m \ln S - m \ln S_0$$

Equation 3 shows that for a flaw distribution characterized by a two-parameter Weibull distribution a plot of $\ln\{(1/L)\ln[1/(1-P)]\}$ against $\ln S$ will be linear with slope $m$. However, from the lack of linearity of such plots for extensive data reported in the literature [2–4] it has been concluded [5] that the single Weibull distribution is inconsistent with experimental data. As an alternative approach, a multimodal Weibull distribution based on the multi-risk model [8] is usually used for the analysis of strength data. For a bimodal distribution the function is given by [9–11]

$$P(S) = 1 - [1 - P_1(S)][1 - P_2(S)]$$
$$= 1 - \exp[-(S/S_{01})^{m_1} - (S/S_{02})^{m_2}] \quad (4)$$

where $P_1(S)$ and $P_2(S)$ are the strength distribution


0022-2461/88 $03.00 + .12 

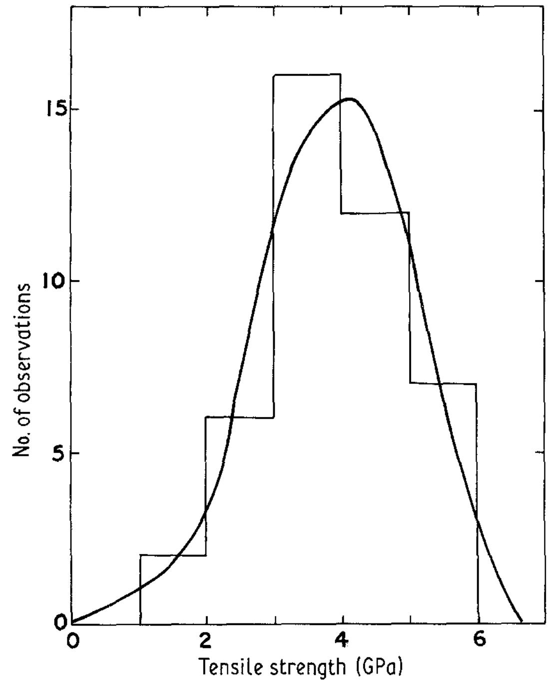


*Figure 1* Tensile strength histogram of coreless silicon carbide fibres (gauge length 10 mm). The full curve corresponds to the fitted beta distribution. $N = 43$.

functions of the defect subpopulation number 1 and 2 respectively. Each of them is described as a single Weibull distribution. It may be mentioned that Olshansky and Maurer [7] have pointed out that a flaw distribution in which the slope of $\ln\{(1/L)\ln[1/(1-P)]\}$ decreases with $\ln S$ cannot be interpreted in terms of two Weibull flaw distributions unless it is assumed that neither distribution extends over the entire experimental range of stresses. Snowden [12] has analysed the statistical justification for using a bimodal Weibull distribution by calculating the standardized coefficients of skewness and kurtosis of various bimodal strength data of optical glass fibres and concluded that the beta distribution, rather than the bimodal Weibull distribution describes the data best. As an example, an analysis of bimodal strength data, reported by Goda and Fukunaga [2], for coreless silicon carbide fibres (Nicalon, produced by Nippon Carbon Co. Ltd.) tested at a gauge length of 10 mm is shown in Fig. 1. The values of the standardized coefficients of skewness square and kurtosis, as calculated from the various moments of this distribution are obtained are 0.038 and 2.53 respectively. Analysis of the data in terms of the Pearson system [13] of a probability-density function yields the beta distribution, which is also shown in Fig. 1.

The beta distribution is characterized by two shape parameters with the value of the variate being limited to a finite interval. This is more realistic for brittle materials like silicon carbide or alumina fibres, with a lower bound of zero being reasonable and an upper bound equal to the theoretical maximum strength $\sim E/10$ [14], where $E$ is the tensile modulus of fibres. On the other hand, the Weibull distributions given by Equations 3 and 4, require $S = \infty$ for certainty of failure ($P = 1$), which is a physically unsatisfactory boundary condition.

To overcome this limitation Kies [15] proposed a modification of the Weibull distribution in the form

$$N(S) = [(S - S_L)/(S_U - S)]^{m_0} \quad (5)$$

where $S_L$ and $S_U$ are the lower and upper limiting strengths respectively and $m_0$ is defined as the damage coefficient. However, it has been shown [16] that even this modified form is not applicable to the entire strength data of brittle materials.

Functionally Equation 5 is similar in form to one obtained from the beta distribution, except it has only one shape parameter. Thus a further modification of Equation 5 is suggested in the form

$$N(S) = [(S - S_L)/S_{01}]^{m_1}\,[S_U - S)/S_{02}]^{m_2} \quad (6)$$

where $S_{01}$, $S_{02}$ and $m_1$, $m_2$ are the two scaling parameters and shape parameters respectively. Equation 6 along with Equation 1 gives

$$\ln\{(1/L)\ln[1/(1-P(S))]\} = m_1 \ln[(S - S_L)/S_{01}] - m_2 \ln[(S_U - S)/S_{02}] \quad (7)$$

## 2.2. Goodness of fit

In order to compare strength data with the distribution function, some measure of goodness of fit is needed. In this study, the sum of squares is used as a measure of the goodness of fit between the function and data. The sum of squares is given by

$$Q = 1 - \sum_{i=1}^{n} (S_i - \hat{S}_i)^2 \Big/ \sum_{i=1}^{n} (S_i - \bar{S})^2 \quad (8)$$

where $\hat{S}_i$ is the value of failure stress calculated for the appropriate $P$ value from the ranking of failure strengths and the calculated parameters of the distribution function; $S_i$ is the measured strength values and $\bar{S}$ is the mean of the distribution. The $\hat{S}_i$ values corresponding to different values of $P$ are obtained from Equation 7 by using Newton–Raphson's method. For a perfect fit $Q = 1$, in general $Q > 0.95$ indicates a good fit.

## 2.3. Prediction of strength distribution and the mean value at different gauge lengths

The strength distribution at different gauge lengths can be predicted from Equations 1 and 6, once the parameters of Equation 6 have been evaluated by fitting Equation 7 to data at one gauge length.

An average strength $\bar{S}$ at gauge length $L$ can be calculated using Equations 1 and 6 as

$$\bar{S} = S_L + \int_{S_L}^{S_U} \exp\{-L[(S - S_L)/S_{01}]^{m_1} / [(S_U - S)/S_{02}]^{m_2}\}\,dS \quad (9)$$

The strength at different gauge lengths can be obtained from Equation 9 by numerical integration of the right-hand side integral.

## 3. Data analysis and discussion

In order to evaluate the applicability of the proposed distribution function the strength data of coreless silicon carbide fibres (Nicalon, produced by Nippon Carbon Co. Ltd.) and alumina fibres (Fibre FP,

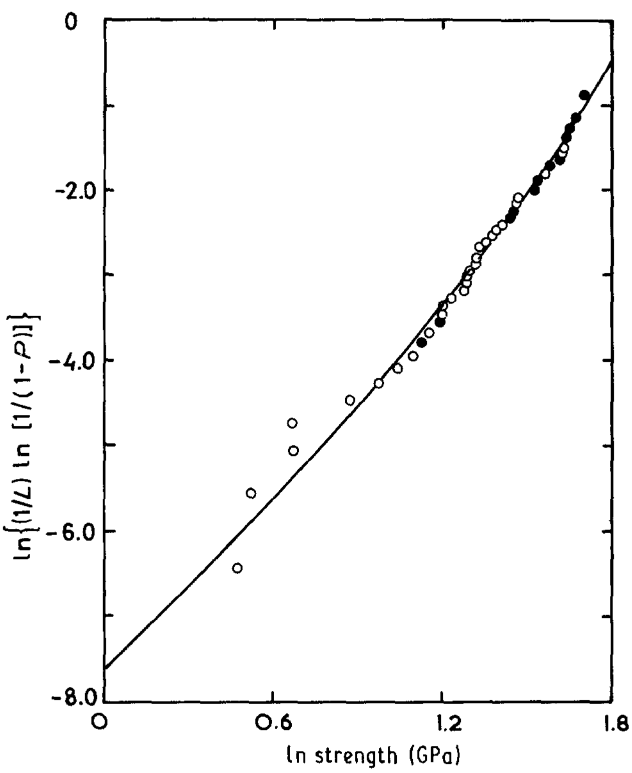


*Figure 2* Weibull plot for SiC fibres and the cumulative distribution curve estimated from Equation 6. (○ surface defect, ● volume defect) Gauge length = 10 mm.

produced by Du Pont), reported by Goda *et al.* [2] and Nunes [17] respectively, have been analysed. Goda *et al.* [2] tested the fibres using gauge lengths of 10 mm and 254 mm (10 in) respectively. In addition, primary fracture surfaces of the fibres were observed by a scanning electron microscope to identify the type of flaw responsible for failure. For silicon carbide (SiC) fibres two main types of defects were identified as "flaw" of "pit" type surface defects and "void" type volume defects. In additon there were undetected defects which were assigned to surface. Goda *et al.* [2] also experimented with alumina ($Al_2O_3$) fibres to identify the kind of fracture mode. However, from the granular appearance of fractured surfaces it was not possible to identify the defect. They attributed two kinds of fracture mode of $Al_2O_3$ fibres to "crooked" and "uncrooked" fibre and concluded that Nune's [17] data shows similar tendency.

Fig. 2 shows Weibull plots for the tensile strength of SiC fibres. Since Equation 7 is not linear in their parameters, an interactive least square method used for finding the parameters using the minimum total variance as the criteria. Initial estimates of $S_U$ and $S_L$ were made from the fitted beta distribution. A set of values were assumed for $S_{01}$ and $S_{02}$ and the values of the parameters $m_1$ and $m_2$ were evaluated from the data by regression analysis. From the calculated and experimental values of $\ln \{(1/L) \times \ln [1/(1 - P(S)]\}$, a least squares sum was evaluated for the set of parameters $S_{01}$ and $S_{02}$. The process was repeated by changing the values of $S_L$ and $S_U$ until the minimum least squares sum was found. It may be noted that the numerator of the second term on the right-hand side of Equation 8 gives the least squares sum. Since for a given set of data the quantity $(S_i - \bar{S})^2$ is fixed, the minimum least square sum will automatically ensure the maximum value of $Q$. A similar analysis was also carried out for $Al_2O_3$ fibres. The data points for this fibre are plotted in Fig. 3. The values of parameters obtained for both the group of fibres are given in Table I along with the $Q$ values calculated from Equation 8. The fitted equations are shown as full lines in the respective figures; in both the proposed equation provides excellent agreement with the data. This is further supported by the $Q$ values of 0.99 and 0.97 obtained for SiC and $Al_2O_3$ fibres respectively. The corresponding values for the bimodal Weibull distribution are estimated to be 0.98 and 0.94 from the parameters given by Goda *et al.* [2], indicating that the proposed distribution describes the data better. It may be noted that in the case of SiC fibres, though the strength values arising out of the volume defect is concentrated in the region of high strength, both the groups contain strength values arising out of both volume and surface defects. Goda and Fukunage [2] justify the application of a bimodal Weibull distribution based on the fracture mode i.e., the volume of surface defect. However, it is clear from Fig. 2, that the strength values are dependent on flaw size rather than on the fracture mode. Application of the bimodal Weibull distribution requires the identification of flaws by fractographic analysis. As pointed out by Jakus *et al.* [10] the determination of the four Weibull parameters necessary to describe a bimodal concurrent flaw population is also not straightforward. To accurately analyse the data, samples that fail from one type of flaw must be included as censored data in the ranking of strengths of other type of flaw and *vice versa.* On the other hand, proposed distribution does not require the flaw source to be identified. This is especially useful for cases where the flaw identification is difficult.

Fig. 4 shows a comparison of theoretically predicted strength values of SiC fibres with those of experimental ones, measured at the gauge lengths of 5 mm and 50 mm. The theoretical values have been calculated from Equations 1 and 6 on the basis of parameters estimated at 10 mm gauge length (Table I). It can be seen from Fig. 4 that there is close agreement between the predicted values and the experimental data. Fig. 5 shows a similar plot for $Al_2O_3$ fibres tested at gauge lengths of 0.5 in. (12.7 mm) and 5 in. (127 mm). Theoretical predictions are based on the parameters evaluated for a gauge length of 10 in. (254 mm). Here again, the predicted values are in close agreement with the experimental data.

The logarithms of the average strengths of SiC and $Al_2O_3$ fibres are shown in Figs 6a to b respectively. Full lines correspond to the average strength curves calculated from Equation 9 on the basis of parameters estimated at gauge lengths of 10 mm and 10 in. respectively. As can be seen from the figures the predicted curve agrees with the experimental data points thereby proving the validity of Equation 6 in describing the strength data.

Like the Weibull modulus $m$, if we associate the parameters $m_1$ and $m_2$ with the scatter in strength data of the high and low strength groups of the distribution ion respectively, it can be seen from Table I,

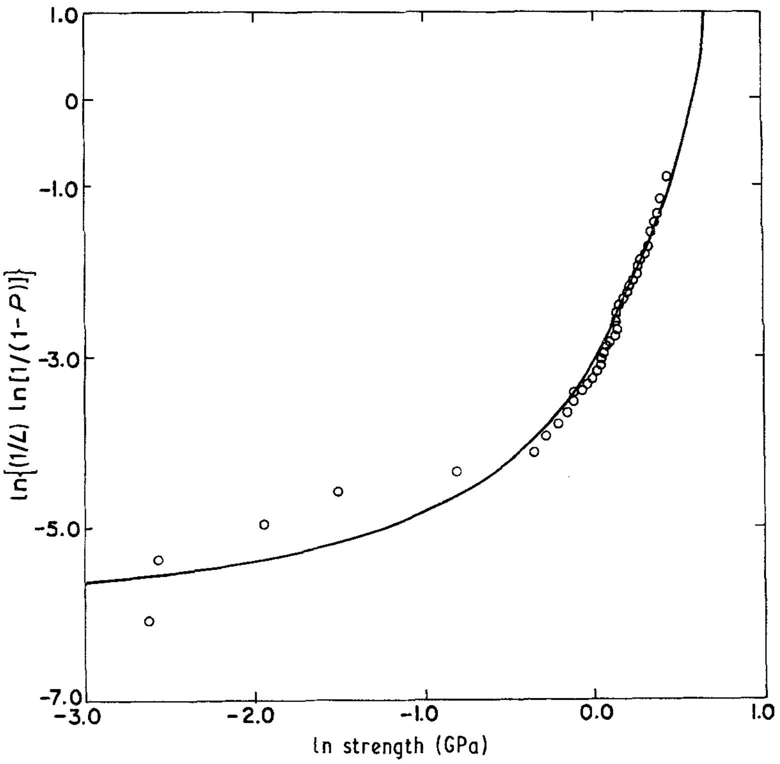


*Figure 3* Weibull plot for $Al_2O_3$ fibres and the cumulative distribution curve estimated from Equation 6. Gauge length = 10 in (254 mm)

TABLE I Values of parameters of equation 6 obtained from failure analysis and the sum of squares

| Fibre type | Gauge length (mm) | $S_U$ (GPa) | $S_L$ (GPa) | $S_{01}$ (GPa) | $S_{02}$ (GPa) | $m_1$ | $m_2$ | $Q$ | Remarks |
|---|---|---|---|---|---|---|---|---|---|
| SiC | 10 | 16.0 | 0 | 2.0 | 4.5 | 2.88 | 4.66 | 0.99 | Ref. [2] |
| $Al_2O_3$ | 254 (10 in.) | 3.0 | 0 | 0.2 | 1.2 | 0.054 | 6.19 | 0.97 | Ref. [17] |

that the low values of $m_1$ in both the cases indicate a larger scatter in the lower strength group. This is consistent with the experimental observations as shown in Figs. 2 and 3.

## 4. Conclusions

The tensile strength data of coreless silicon carbide and alumina fibres have been analysed in terms of a new modified Weibull distribution function using the weakest-link theory. The strength–length relationship has also been derived. The specific conclusions that can be drawn from this study are as follows

(1) The flaw distribution given by Equation 6 can be used to describe the strength distribution in brittle fibres like coreless silicon carbide or alumina. The function provides an upper and lower limiting strength

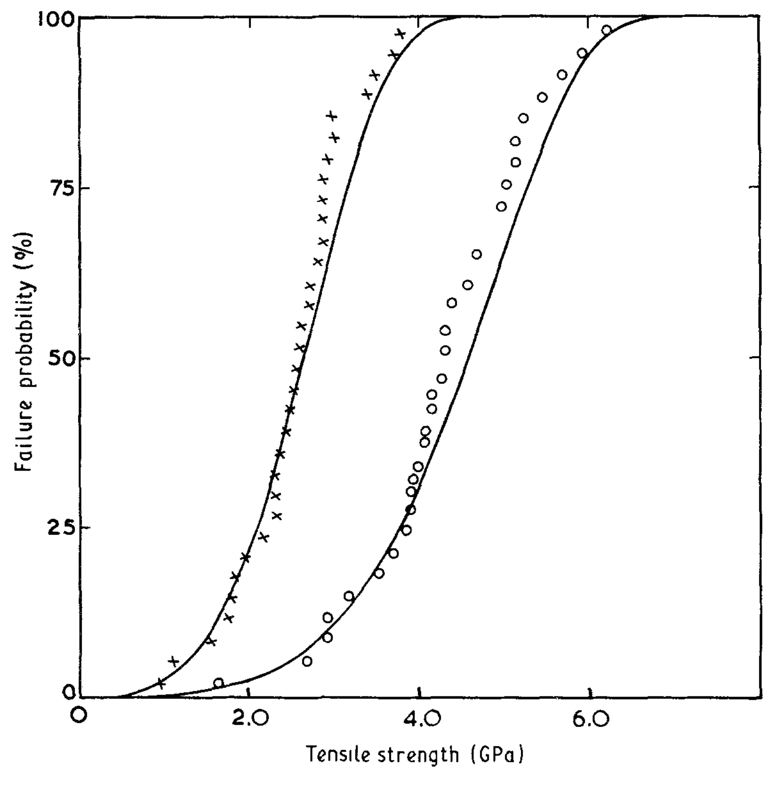


*Figure 4* Predicted (——) failure probability of SiC fibres at gauge lengths of 5 mm (O) and 50 mm (x) compared with the experimental data points.

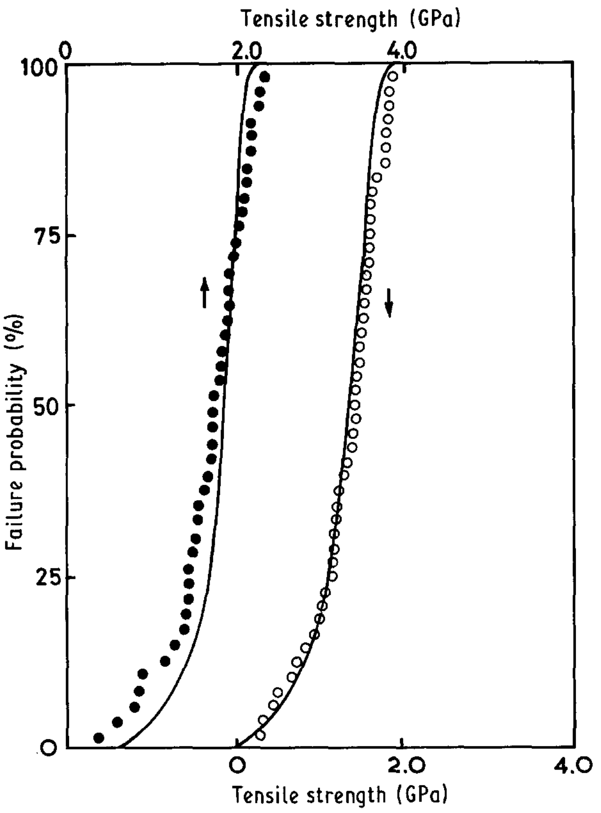


*Figure 5* Predicted failure propability of $Al_2O_3$ fibres at gauge lengths of 0.5 in. (12.7 mm) (●) and 5 in. (127 mm) (○), compared with the experimental data points.

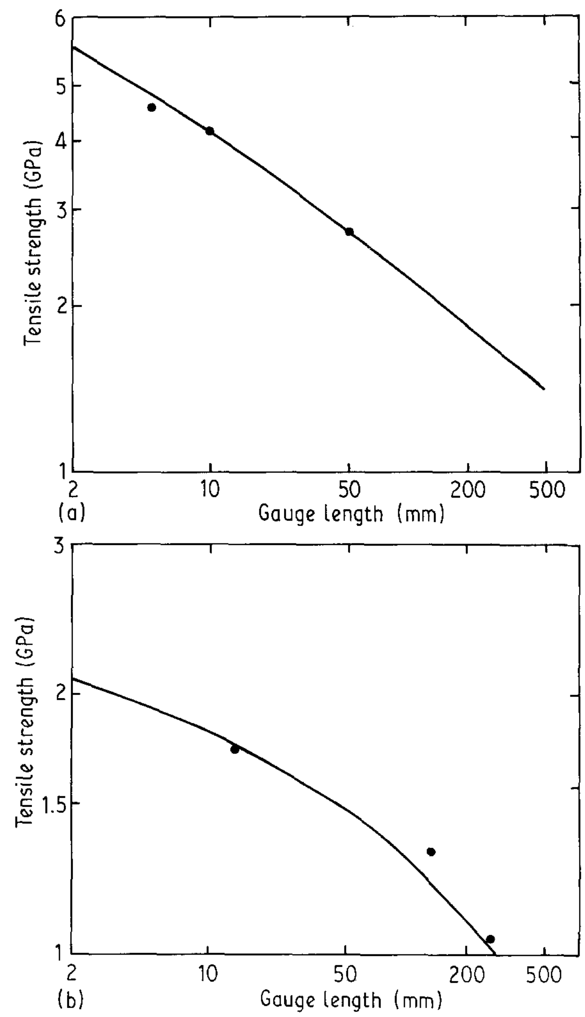


*Figure 6* Average strength plots of SiC (a) and (b) $Al_2O_3$ fibres. The full line corresponds to Equation 9.

and is consistent with the boundary conditions of the physical phenomena it represents.

(2) The strength distribution and the average values at different gauge lengths, predicted from this function show close agreement with the experimental data points.

## Acknowledgements
The author thanks Dr S. Kumar, Director of the Institute for his kind permission to publish this paper. Thanks are also due to Shri S. Sengupta for the preparation of the line diagrams.

## References
1. W. WEIBULL, *J. Appl. Mech.* **18** (1951) 293.
2. K. GODA and H. FUKUNAGA, *J. Mater. Sci.* **21** (1986) 4475.
3. P. MARTINEAU, M. LAHAYE, R. PAILLER, R. NASLAIN, M. COUZI and F. CRUEGE, *ibid.* **19** (1984) 2731.
4. J. W. HITCHON and D. C. PHILLIPS, *Fibre Sci. Tech.* **12** (1979) 217.
5. B. F. JONES and B. J. S. WILKINS, *ibid.* **5** (1972) 315.
6. O. M. COTCHICK, R. C. HINK and R. E. TRESSLER, *J. Compos. Mater.* **9** (1975) 327.
7. R. OLSHANSKY and R. D. MAURER, *J. Appl. Phys.* **4** (1976) 4497.
8. R. J. HERMAN and R. K. N. PATELL, *Technometics* **13** (1971) 385.
9. T. E. EASLER, R. C. BRADT and R. E. TRESSLER, *J. Amer. Ceram. Soc.* **64** (1981) C-53.
10. K. JAKUS, J. E. RITTER Jr., T. SERVICE and D. SONDERMAN, *ibid.* **64** (1981) C-174.
11. D. SONDERMAN, K. JAKUS, J. E. RITTER Jr., S. YUHASKI Jr. and T. H. SERVICE, *J. Mater. Sci.* **20** (1985) 207.
12. W. E. SNOWDEN, in "Fracture of Ceramics", Vol. 4, edited by R. C. Bradt, D. P. H. Hasselman and F. F. Lange, (Plenum, New York, 1978) pp. 143–59.
13. J. H. POLLARD, "A handbook of Numerical and Statistical Techniques", (Cambridge University Press, Cambridge, 1977) ch. 11.
14. R. HOUWINK and H. K. DeDECKER, "Elasticity, Plasticity and Structure of Matter" (Cambridge University Press, Cambridge, 1971).
15. J. A. KIES, "The Strength of Glass", NRL Report 5093, Naval Research Lab., Washington, DC (1958).
16. D. KALISH, B. K. TARIYAL and R. O. PICKWICK, *Amer. Ceram. Soc. Bull.* **56** (1977) 451.
17. J. NUNES, AMMRC TR 82-61 (Army Materials and Mechanics Research Centre, Watertown, MA, 1982) p. 1.
18. R. OLSHAUSKY and R. D. MAURER, *J. Appl. Phys.* **47** (1976) 4497.

*Received 16 June*
*and accepted 16 December 1987*